

Effective action for fermions with anomalous magnetic moment from Foldy-Wouthuysen transformation.

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Abstract

In this paper we calculate the effective action for neutral particles with anomalous magnetic moment in an external magnetic and electric field. We show that we can take advantage from the Foldy Wouthuysen transformation for such systems, determined in our previous works: indeed, by this transformation we have explicitly evaluated the diagonalized Hamiltonian, allowing to present a closed form for the corresponding effective action and for the partition function at finite temperature from which the thermodynamical potentials can be calculated.

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1 Introduction.

In some recent works [1–3] we have used Grassmann variables in order to investigate the properties of the interactions of pseudo-classical spinning relativistic particles and superparticles [4–7] with electromagnetic couplings. The description of spinning particles was generalized by allowing for the presence of an anomalous magnetic moment (in the following: a.m.m.) [8–11] and the Foldy-Wouthuysen transformation (hereafter FWT) was fruitfully applied to a pseudo-classical spinning particle with a.m.m. in a classical stationary electromagnetic field [2, 12–15]. We studied, in particular, the cases in which the result could be expressed in a closed form. These turned out to be the systems of a neutral particle in a stationary electric field, a neutral

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particle in a stationary magnetic field and a charged particle in a stationary magnetic field. An important technical point that made it possible to obtain the results was the possibility of exploiting two different representations of the Clifford algebra realizing the Dirac brackets needed for the quantization of the pseudo-classical variables and intertwined by a Pauli-Gursey unitary transformation, that allow us to get more easily the exact form for the three different interacting cases. Finally, in a later work [3], a Schwinger proper time description of these systems has also been given.

The purpose of this work is to study the finite temperature effective action of neutral spinning relativistic particles with a.m.m. in external stationary uniform magnetic and electric fields. Using our previous results on the FWT, we will determine the effective action and the corresponding thermodynamical potentials in closed form.

2 The method.

In order to illustrate the method we recall that the partition function for a spinor particle in an electromagnetic field can be written in the functional integral form as

$$\begin{aligned} Z[A, \eta, \bar{\eta}] &= \int \mathcal{D}[\bar{\psi}(x)] \mathcal{D}[\psi(x)] \exp \left\{ i \int d^4x \left[\bar{\psi}(x)(i\hat{\partial} - q\hat{A}(x) - m) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x) \right] \right\} \\ &= \exp \left\{ -i \int d^4x \int d^4y \bar{\eta}(x) S_F(x, y) \eta(y) \right\} \cdot \det[-iS_F^{-1}] \end{aligned} \quad (2.1)$$

where $\bar{\eta}(x)$, $\eta(x)$ are the fermionic sources for the fields $\psi(x)$, $\bar{\psi}(x)$ respectively and $S_F = (i\hat{\partial} - q\hat{A}(x) - m)^{-1}$ is the Green's function for the Dirac operator. Obviously the quadratic functional integral in (2.1) has been done with the rules of the fermionic case. We simply denote by $Z[A]$ the partition function with vanishing sources: the effective action is then given by $-i$ the times logarithm of $Z[A]$, namely

$$S_{\text{eff}}[A] = -i \ln Z[A] = -i \ln \det (i\hat{\partial} - q\hat{A}(x) - m) = -i \text{tr} \ln (i\hat{\partial} - q\hat{A}(x) - m) \quad (2.2)$$

In eq. (2.2) the determinant and the trace have to be taken both in the space of the coordinates and in the space of the Dirac variables.

We will show how the FWT can be used for obtaining the explicit form of the effective action, by sketching the simplest case of a free Dirac particle and of a Dirac particle in a static and uniform magnetic field.

(i) The free spinor particle. Let first $A^\mu(x) = 0$. As $\gamma^0 \equiv \beta = \text{diag}(+1, +1, -1, -1)$ and therefore $\det(\gamma^0) = 1$, we have to calculate

$$S_{\text{eff}}[0] = -i \ln \det (p_0 - (\vec{\alpha} \cdot \vec{p} + \beta m)), \quad p_0 = i\partial/\partial t, \quad \vec{p} = -i\vec{\nabla}. \quad (2.3)$$

If we now recall that the FWT for the free case is generated by

$$U_{FW} = \exp \left(\frac{\arctan(|\vec{p}|/m)}{2|\vec{p}|} \beta \vec{\alpha} \cdot \vec{p} \right) \quad (2.4)$$

and we observe that U_{FW} commutes with p_0 , we can write:

$$S_{\text{eff}}[0] = -i \ln \det \left(p_0 - U_{FW} (\vec{\alpha} \cdot \vec{p} + \beta m) U_{FW}^{-1} \right) = -i \ln \det \left(p_0 - \beta \sqrt{\vec{p}^2 + m^2} \right) \quad (2.5)$$

Due to the diagonal form of β , the determinant in the Dirac space is immediate and we are reduced to calculating

$$S_{\text{eff}}[0] = -i \ln \det \left[\left(i \frac{\partial}{\partial t} - \sqrt{-\vec{\nabla}^2 + m^2} \right) \left(i \frac{\partial}{\partial t} + \sqrt{-\vec{\nabla}^2 + m^2} \right) \right]^2 \quad (2.6)$$

where now the determinant is only the functional determinant in the coordinate space. Hence we obtain the well known result [16, 18]

$$S_{\text{eff}}[0] = -i \text{tr} \ln \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2 \right)^2 = -2i \text{tr} \ln (\square + m^2). \quad (2.7)$$

The last expression of (2.7) is calculated, as usual, by inserting complete sets of momentum eigenstates. Taking into account of the infinite volume factor coming from the scalar product $\langle p | p \rangle = (2\pi)^4 \delta^4(0) = \int d^4x \equiv V_4$, we have

$$S_{\text{eff}}[0] = -2i V_4 \int \frac{d^4p}{(2\pi)^4} \ln(-p^2 + m^2) \quad (2.8)$$

Finally, recalling that the effective action is the space-time integral of the effective Lagrangian density \mathcal{L}_{eff} , we can integrate over p_0 and write the effective potential of our system as

$$V_{\text{eff}}[0] = -\mathcal{L}_{\text{eff}}[0] = -2 \int \frac{d^3p}{(2\pi)^3} \sqrt{\vec{p}^2 + m^2} = \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} \exp(-m^2 s) \quad (2.9)$$

where the last expression is a straightforward consequence of the well known representation of the logarithm of an operator

$$\ln A = - \int_0^\infty \frac{ds}{s} \exp(-is(A - i\epsilon)) \quad (\epsilon > 0), \quad (2.10)$$

when applied to $A = \square + m^2$.

(ii) The spinor particle in a uniform magnetostatic field. The next example we want to describe is that of a Dirac particle in a static and uniform magnetic field. We will assume a vector potential and a corresponding magnetic field given by

$$A^0 = 0, \quad \vec{A} = (0, Bx, 0), \quad \vec{B} = (0, 0, B). \quad (2.11)$$

Along the same lines as for the free case, we see that equation (2.3) must now obviously changed into

$$S_{\text{eff}}[A] = -i \ln \det (p_0 - (\vec{\alpha} \cdot \vec{\pi} + \beta m)), \quad \vec{\pi} = \vec{p} - q\vec{A} \quad (2.12)$$

where q is the particle electric charge. The explicit form of the FWT generator is now more complex, namely [12, 17]

$$U_{FW} = \exp \left(\beta \mathcal{O} \phi \right), \quad \mathcal{O} = \vec{\alpha} \cdot \vec{\pi}, \quad \mathcal{O}^2 = \vec{\pi}^2 - q \vec{\Sigma} \cdot \vec{B} \quad (2.13)$$

where the angle ϕ is given by

$$\phi = 1/(2\sqrt{\mathcal{O}^2}) \arctan(\sqrt{\mathcal{O}^2}/m) \quad (2.14)$$

By applying the FWT we find now

$$S_{\text{eff}}[A] = -i \ln \det \left(p_0 - U_{FW} \mathcal{O} U_{FW}^{-1} \right) = -i \ln \det (p_0 - \beta \sqrt{\mathcal{O}^2 + m^2}) \quad (2.15)$$

Using commutation and anti-commutation relations

$$[\gamma_5, \vec{\Sigma}] = [\gamma_5, \beta]_+ = 0, \quad (2.16)$$

we have

$$\begin{aligned} \det (p_0 - \beta \sqrt{\mathcal{O}^2 + m^2}) &= \det (\gamma_5 (p_0 - \beta \sqrt{\mathcal{O}^2 + m^2}) \gamma_5) = \det (p_0 + \beta \sqrt{\mathcal{O}^2 + m^2}) = \\ &= \sqrt{\det (p_0 - \beta \sqrt{\mathcal{O}^2 + m^2}) \det (p_0 + \beta \sqrt{\mathcal{O}^2 + m^2})} = \sqrt{\det (p_0^2 - \vec{\pi}^2 - m^2 + q \vec{\Sigma} \cdot \vec{B})} \end{aligned} \quad (2.17)$$

We can now observe that the matrix operator under the last square root is in the diagonal block form $q \vec{\Sigma} \cdot \vec{B} = q B \text{diag}(\sigma_3, \sigma_3)$, so that the Dirac space determinant is easily evaluated and the effective action reads

$$S_{\text{eff}}[A] = -i \ln \det \left((-p_0^2 + \vec{\pi}^2 + m^2 - qB) (-p_0^2 + \vec{\pi}^2 + m^2 + qB) \right) \quad (2.18)$$

where, again, we are left with the functional determinant on the space-time variables. Substituting the logarithm of the determinant with the trace of the logarithm we see that we are substantially reduced to the scalar case and we have to evaluate the quantity

$$J_{\pm} = -i \text{tr} \ln (-\pi^2 + M_{\pm}^2), \quad M_{\pm}^2 = m^2 \pm qB, \quad \pi^\mu = p^\mu - qA^\mu \quad (2.19)$$

It is well known that there are several ways to calculate the above trace. We can use the knowledge of the eigenfunctions of the operator $-\pi^2 + M_{\pm}^2$, or we can solve the Heisenberg equation of motion and find the evolution operator in the Schwinger proper time formalism, or else we can calculate the evolution operator by means of a path integral [11], which is possible as we have reduced the problem in a quadratic Bose form. We notice that for a constant magnetic field we could have proceeded also directly by integrating over the odd variables; since this procedure cannot be directly exported in the presence of an anomalous magnetic moment, we show how the other methods, which generally apply, will be used.

Specifying equation (2.10) to our present case, we write

$$J_{\pm} = i \operatorname{tr} \int_0^{\infty} \frac{ds}{s} \exp\left(-is(-\pi^2 + M_{\pm}^2 - i\epsilon)\right) = -i \operatorname{tr} \int_{-\infty}^0 \frac{ds}{s} \exp\left(-is(\pi^2 - M_{\pm}^2 + i\epsilon)\right) \quad (2.20)$$

The explicit form of the wave operator that appears in (2.20) is

$$-(\pi^2 - M_{\pm}^2) = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + 2iqBx \frac{\partial}{\partial y} + q^2 B^2 x^2 + M_{\pm}^2 \quad (2.21)$$

and we denote its eigenfunctions by $|p_0, p_y, p_z; n\rangle \equiv |p_0, p_y, p_z\rangle |\varphi_n\rangle$. The first term in the right hand side is the factor corresponding to the conserved momenta p_0, p_y, p_z , while $\varphi_n(x) = \langle x | \varphi_n \rangle$ solves the eigenvalue problem

$$\left[-\frac{\partial^2}{\partial x^2} + q^2 B^2 \left(x - \frac{p_y}{qB}\right)^2\right] \varphi_n(x) = \lambda_n \varphi_n(x) \quad (2.22)$$

with $\lambda_n = (2n+1)qB$, being the eigenvalues for a harmonic oscillator with frequency qB . We assume the functions $\varphi_n(x)$ normalized to unity. Inserting the appropriate completeness relations the conserved momenta give rise to an infinite factor $(2\pi)^3 \delta^3(0) = L_2 L_3 L_0$, where L_2, L_3 refer to the space directions y and z and L_0 to time. After some lengthy but straightforward calculations, the quantity J of equation (2.20) becomes

$$\begin{aligned} J_{\pm} &= i L_2 L_3 L_0 \int_0^{\infty} \frac{ds}{s} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \int_0^{qBL_1} \frac{dp_y}{2\pi} \\ &\quad \sum_{n=0}^{\infty} \exp\left[-is\left(-p_0^2 + p_z^2 + M_{\pm}^2 + (2n+1)qB - i\epsilon\right)\right] \\ &= i \frac{L_1 L_2 L_3 L_0}{8\pi^2} \int_0^{\infty} \frac{ds}{s^3} (sqB) \exp(\mp isqB) \sum_{n=0}^{\infty} \exp\left(-is((2n+1)qB + m^2 - i\epsilon)\right) \end{aligned} \quad (2.23)$$

where the integration in p_y reconstructs the four dimensional volume V_4 and the contribution of the p_0 and p_z integrations is, respectively, $\exp(\pm i\pi/4) (\pi/s)^{1/2}$. In order to find the effective action we must add the contributions of M_+ and M_- : after summing the series we find

$$S_{\text{eff}}[A] = J_+ + J_- = \frac{L_1 L_2 L_3 L_0}{8\pi^2} \int_0^{\infty} \frac{ds}{s^3} \exp(-is(m^2 - i\epsilon)) (sqB) \cot(sqB) \quad (2.24)$$

Some care is in order for the calculation of this integral. In the first place because of the poles of $\cot(sqB)$ on the real axis. Secondly, because of possible counterterms that should be added in (2.24) in order to renormalize the effective Lagrangian $\mathcal{L}_{\text{eff}}[A] = S_{\text{eff}}[A]/V_4$. The problem of the poles of the integrand on the real axis is solved by observing that the integral converges for $\operatorname{Im}(s) < 0$, so that the integration path can be deformed from the positive real semi-axis to the negative imaginary semi-axis, $s \mapsto -is$. We can next normalize to unity the partition function for a vanishing field. This amounts to considering $Z[A]/Z[0]$ instead of $Z[A]$. On the

effective action this normalization has the effect of subtracting the free part from the result in the presence of a magnetic field, so that $S_{\text{eff}}[A = 0] = 0$. After these two steps we get the effective Lagrangian

$$\mathcal{L}_{\text{eff}}[A] = -\frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} \exp(-sm^2) (s q B) (\coth(s q B) - 1) \quad (2.25)$$

We easily see that the integral in (2.25) is still divergent at $s = 0$ and needs renormalization. We thus assume the charge q and the magnetic field B of (2.25) as bare quantities q_0, B_0 ; we consider the expansion $(s q_0 B_0) (\coth(s q_0 B_0) - 1) = (1/3) (s q_0 B_0)^2 + O(s^4)$; we add to $\mathcal{L}_{\text{eff}}[A]$ the contribution of the tree level, namely $-(1/2)B_0^2$; we rewrite (2.25) by adding and subtracting the divergent part. We finally obtain

$$\begin{aligned} \mathcal{L}_{\text{eff}}[A] = & -\frac{1}{2} B_0^2 \left(1 + \frac{1}{12\pi^2} q_0^2 \int_0^\infty \frac{ds}{s} \exp(-sm^2) \right) \\ & - \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} \exp(-sm^2) (s q_0 B_0) \left(\coth(s q_0 B_0) - \frac{1}{3} (s q_0 B_0)^2 - 1 \right) \end{aligned} \quad (2.26)$$

We now introduce the renormalization constant for the wave function, the renormalized charge and magnetic field:

$$Z_3^{-1} = 1 + \frac{1}{12\pi^2} q_0^2 \int_0^\infty \frac{ds}{s} \exp(-sm^2), \quad q = q_0 Z_3^{\frac{1}{2}}, \quad B = B_0 Z_3^{-\frac{1}{2}}. \quad (2.27)$$

We have $q_0 B_0 = q B$ and we can write the final expression of the renormalized effective action [19]

$$\mathcal{L}_{\text{eff}}[A] = -\frac{1}{2} B^2 - \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} \exp(-sm^2) (s q B) \left(\coth(s q B) - \frac{1}{3} (s q B)^2 - 1 \right) \quad (2.28)$$

3 Effective action for neutral fermions with a.m.m. in static magnetic and electric fields.

Let us recall the Hamiltonian form of the Dirac equation for a fermion with a.m.m. μ interacting with an external electromagnetic field:

$$i \frac{\partial \psi(t, \vec{x})}{\partial t} = H_D \psi(t, \vec{x}), \quad H_D = \vec{\alpha} \cdot \vec{\pi} + q A_0 + \beta m + \frac{e\mu}{8m} \sigma_{\rho\nu} F^{\rho\nu}, \quad (3.1)$$

where

$$\sigma^{\rho\nu} = \frac{i}{2} [\gamma^\rho, \gamma^\nu], \quad F^{\rho\nu}(x) = \frac{\partial A^\nu}{\partial x_\rho} - \frac{\partial A^\rho}{\partial x_\nu}, \quad (3.2)$$

and, as usual, $\vec{\pi} = \vec{p} - q\vec{A}$.

In addition to (3.1) we can also use an equivalent expression for the wave equation, obtained by means of the Pauli-Gursey unitary transformation generated by applying $\exp[i(\pi/4)\gamma_5]$ to the spinor wave function [2, 11, 20] and, correspondingly, by mapping the Hamiltonian H_D into the Hamiltonian $H_{PG} = \exp[i(\pi/4)\gamma_5] H_D \exp[-i(\pi/4)\gamma_5]$. By substituting $F^{\rho\nu}$ by its explicit expression in terms of the electric and the magnetic field, we can write H_D and H_{PG} as

$$\begin{aligned} H_D &= \vec{\alpha} \cdot \vec{\pi} + qA_0 + \beta m + \frac{ie\mu}{4m} \beta \vec{\alpha} \cdot \vec{E} - \frac{e\mu}{4m} \beta \vec{\Sigma} \cdot \vec{B} \\ H_{PG} &= \vec{\alpha} \cdot \vec{\pi} + qA_0 - i\beta\gamma_5 m + \frac{e\mu}{4m} \beta\gamma_5 \vec{\alpha} \cdot \vec{E} + \frac{ie\mu}{4m} \beta\gamma_5 \vec{\Sigma} \cdot \vec{B} \end{aligned} \quad (3.3)$$

where $\vec{\Sigma} = \gamma_5 \vec{\alpha}$. The effective action can be equivalently obtained by the Dirac or the Pauli-Gursey representation and reads

$$S_{\text{eff}}[A] = -i \ln \det \left(i \frac{\partial}{\partial t} - H_D \right) = -i \ln \det \left(i \frac{\partial}{\partial t} - H_{PG} \right) \quad (3.4)$$

where H_D and H_{PG} are given in (3.3).

Since we want to calculate the effective action by using the FWT along the lines described in Section 2, we find it useful to recall some results determined in [2] and related to the cases in which the FWT can be found in closed form, namely the cases of a neutral fermion with a.m.m. in a static and uniform magnetic field or in a static and uniform electric field. Letting first $q = 0$ and $\vec{E} = 0$ in the Hamiltonian H_{PG} in (3.3) and applying to it a unitary transformation generated by

$$U = \exp(\beta \mathcal{O} \phi), \quad \mathcal{O} = H_{PG}, \quad \phi = \frac{\pi}{4} (\mathcal{O}^2)^{-\frac{1}{2}} \quad (3.5)$$

we obtain a transformed even Hamiltonian of the form

$$\tilde{H}_{PG} = U H_{PG} U^{-1} = \beta \left[\vec{p}^2 + m^2 + \left(\frac{e\mu}{4m} \right)^2 \vec{B}^2 - \frac{e\mu}{2} \vec{\Sigma} \cdot \vec{B} - \frac{e\mu}{4m} \beta \vec{\Sigma} \cdot (\vec{B} \times \vec{p} - \vec{p} \times \vec{B}) \right]^{\frac{1}{2}} \quad (3.6)$$

Since $(\vec{\Sigma} \cdot \vec{B})^2 = B^2 I$, letting $\vec{p} = 0$ in (3.6), we immediately see that the threshold levels are given by $E_0 = \pm(m \pm (e\mu/4m)B)$ with an energy gap between the intermediate levels

$$\Delta E_0 = 2 \left(m - \frac{|e\mu|}{4m} B \right) \quad (3.7)$$

Introducing the momentum components longitudinal and transverse to the magnetic field \vec{B} ,

$$\vec{p}_\ell = (\vec{p} \cdot \vec{B}) \vec{B} / B^2, \quad \vec{p}_t = \vec{p} - \vec{p}_\ell \quad (3.8)$$

from the dispersion relation (3.6) we can write the energy eigenvalues

$$p_0 = E = \pm \left[p_\ell^2 + \left((p_t^2 + m^2)^{\frac{1}{2}} \pm \frac{e\mu B}{4m} \right)^2 \right]^{\frac{1}{2}} \quad (3.9)$$

In a completely analogous way, if in (3.3) we consider H_D with $q = 0$, $\vec{B} = 0$ and we make a unitary transformation generated by

$$U = \exp(\beta \mathcal{O} \phi), \quad \mathcal{O} = \vec{\alpha} \cdot \left(\vec{p} - \frac{ie\mu}{4m} \beta \vec{E} \right), \quad \phi = \frac{1}{2\sqrt{\mathcal{O}^2}} \arctan\left(\frac{\sqrt{\mathcal{O}^2}}{m}\right) \quad (3.10)$$

we obtain a transformed even Hamiltonian of the form

$$\tilde{H}_D = U H_D U^{-1} = \beta \left[\vec{p}^2 + m^2 + \left(\frac{e\mu}{4m} \right)^2 \vec{E}^2 - \frac{e\mu}{4m} \beta \vec{\nabla} \cdot \vec{E} - \frac{e\mu}{4m} \beta \vec{\Sigma} \cdot (\vec{E} \times \vec{p} - \vec{p} \times \vec{E}) \right]^{\frac{1}{2}} \quad (3.11)$$

The threshold energy levels are now $E_0 = \pm(m^2 + (e\mu/4m)E)^2)^{\frac{1}{2}}$ with energy gap

$$\Delta E_0 = 2 \left(m^2 + \left(\frac{e\mu}{4m} \right)^2 E^2 \right)^{\frac{1}{2}} \quad (3.12)$$

and energy eigenvalues

$$p_0 = E = \pm \left[p_\ell^2 + m^2 + \left(p_t \pm \frac{e\mu}{4m} E \right)^2 \right]^{\frac{1}{2}} \quad (3.13)$$

where now \vec{p}_ℓ and \vec{p}_t are defined by substituting \vec{E} to \vec{B} in (3.8).

Let us examine in more details the two cases.

(i) The spinor with a.m.m. in a uniform magnetostatic field. We want to calculate the functional determinant (3.4). In the momentum representation, after the appropriate FWT the determinant reads

$$I = \det(p_0 - \tilde{H}_{PG}) \quad (3.14)$$

where \tilde{H}_{PG} is given in (3.6).

We find it convenient to introduce the following notation:

$$Q_1 = p_0, \quad Q_2 = \vec{p}^2 + m^2 + \left(\frac{e\mu\vec{B}}{4m} \right)^2, \quad Q_3 = \frac{e\mu}{2} \vec{\Sigma} \cdot \vec{B} + \frac{e\mu}{2m} \beta \vec{\Sigma} \cdot (\vec{B} \times \vec{p}), \quad (3.15)$$

so that we want to find $\det Q$ where

$$Q = Q_1 - \beta \sqrt{Q_2 - Q_3} \quad \text{with} \quad [V_i, V_j] = 0, \quad i, j = 1, 2, 3. \quad (3.16)$$

We therefore need, in the first place, the eigenvalues of V in the spinor space. Observing that

$$Q_3^2 = \left(\frac{e\mu}{2} \right)^2 \left[\vec{B}^2 + \frac{1}{m^2} \left(\vec{B}^2 \vec{p}^2 - (\vec{p} \cdot \vec{B})^2 \right) \right] \equiv \lambda_{3B} I \quad (3.17)$$

we see that the eigenvalues of Q_3 are $\pm\sqrt{\lambda_{3B}}$ each one with multiplicity two. We then follow the same procedure we used in the previous section and recalling the relations (2.16) we write

$$(\det Q)^2 = \det \left[\left(Q_1 - \beta \sqrt{Q_2 - Q_3} \right) \left(\gamma_5 (Q_1 - \beta \sqrt{Q_2 - Q_3}) \gamma_5 \right) \right] = \det \left(Q_1^2 - (Q_2 - Q_3) \right) \quad (3.18)$$

Therefore, taking into account the multiplicity of the eigenvalues, we have that the determinant in the spinor space reads

$$\det Q = \prod_{\eta=\pm 1} \left[p_0^2 - \vec{p}^2 - m^2 - \left(\frac{e\mu\vec{B}}{4m} \right)^2 - \eta \left(\frac{e\mu}{2} \right) \sqrt{\vec{B}^2 + \frac{1}{m^2}(\vec{B}^2 \vec{p}^2 - (\vec{p} \cdot \vec{B})^2)} \right] \quad (3.19)$$

Introducing now the momentum components longitudinal and transverse to the magnetic field \vec{B} ,

$$\vec{p}_\ell = (\vec{p} \cdot \vec{B}) \vec{B} / \vec{B}^2, \quad \vec{p}_t = \vec{p} - \vec{p}_\ell \quad (3.20)$$

the effective action reduces to the following form

$$S_{\text{eff}}[A] = -i \ln \det \left[\prod_{\eta=\pm 1} \left(p_0^2 - \vec{p}_\ell^2 - \left(\sqrt{\vec{p}_t^2 + m^2} + \eta \frac{e\mu B}{4m} \right)^2 \right) \right]. \quad (3.21)$$

By means of the usual identity $\ln \det = \text{tr} \ln$ and by the representation (2.10) for the logarithm, we can write

$$S_{\text{eff}}[A] = iV_4 \sum_{\eta=\pm 1} \int \frac{d^4 p}{(2\pi)^4} \int_0^\infty \frac{ds}{s} \exp \left[-is \left(-p_0^2 + p_\ell^2 + \left((p_t^2 + m^2)^{\frac{1}{2}} + \eta(e\mu/4m) B \right)^2 - i\epsilon \right) \right] \quad (3.22)$$

Calculating the integrals over the zero and parallel components of the momentum, making the complex rotation $s \rightarrow -is$ and dividing by the four dimensional infinite volume V_4 , we find an effective Lagrangian

$$\mathcal{L}_{\text{eff}} = -\frac{1}{8\pi^2} \sum_{\eta=\pm 1} \int_0^\infty \frac{ds}{s^2} \int_0^\infty dp_t p_t \exp \left[-s \left((p_t^2 + m^2)^{\frac{1}{2}} + \eta(e\mu/4m) B \right)^2 \right] \quad (3.23)$$

Let us now define

$$\gamma_B = (|e\mu| B)/(4m^2), \quad z = m^2 s, \quad (3.24)$$

and, accordingly, change the variables in (3.23). We then normalize the effective Lagrangian by subtracting the contribution for a vanishing magnetic field, so that $\mathcal{L}_{\text{eff}}[B=0] = 0$ and we add the free contribution of the magnetic field. After some lengthy but straightforward calculations we then have

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & -\frac{1}{2}B^2 - \frac{m^4}{16\pi^2} \int_0^\infty \frac{dz}{z^3} \left[\exp(-(\gamma_B + 1)^2 z) + \exp(-(\gamma_B - 1)^2 z) - 2 \exp(-z) \right] \\ & - \frac{m^4 \gamma_B}{16\pi^2} \int_0^\infty \frac{dz}{z^2} \int_{-1}^1 du \left[(\gamma_B + 1) \exp(-(\gamma_B + 1)^2 u^2 z) + (\gamma_B - 1) \exp(-(\gamma_B - 1)^2 u^2 z) \right] \end{aligned} \quad (3.25)$$

We finally consider the renormalization of the effective Lagrangian. Indeed the expression (3.25) is still singular for $z = 0$ and we will rewrite it by subtracting the leading divergent terms obtaining a minimally regularized effective Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{eff}} = & -\frac{1}{2}B^2 - \frac{m^4}{16\pi^2} \int_0^\infty \frac{dz}{z^3} \left[\exp(-(\gamma_B + 1)^2 z) + \exp(-(\gamma_B - 1)^2 z) - 2 \exp(-z) \right. \\ & + \gamma_B z \int_{-1}^1 du \left[(\gamma_B + 1) \exp(-(\gamma_B + 1)^2 u^2 z) + (\gamma_B - 1) \exp(-(\gamma_B - 1)^2 u^2 z) \right] \\ & \left. + \exp(-z) \left(-2 \gamma_B^2 z - \gamma_B^2 \left(4 - \frac{1}{3} \gamma_B^2 \right) z^2 \right) \right] + f.t.\end{aligned}\quad (3.26)$$

where *f.t.* indicates the presence of possible finite terms to be determined by a specific renormalization prescription.

(ii) The spinor with a.m.m. in an electrostatic field. We want now to calculate the determinant

$$I = \det(p_0 - \tilde{H}_D) \quad (3.27)$$

with \tilde{H}_D given in (3.11). Since we are considering uniform fields we have the relations

$$\vec{\nabla} \cdot \vec{E} = 0, \quad \vec{E} \times \vec{p} = -\vec{p} \times \vec{E}, \quad (3.28)$$

so that, letting

$$Q_1 = p_0, \quad Q_2 = \vec{p}^2 + m^2 + \left(\frac{e\mu \vec{E}}{4m} \right)^2, \quad Q_3 = \frac{e\mu}{2m} \beta \vec{\Sigma} \cdot (\vec{E} \times \vec{p}), \quad (3.29)$$

we have to evaluate $\det Q$ where Q is again given by (3.16) with the definitions (3.29). We now see that

$$Q_3^2 = \left(\frac{e\mu}{2m} \right)^2 \left(\vec{E}^2 \vec{p}^2 - (\vec{p} \cdot \vec{E})^2 \right) \equiv \lambda_{3E} I \quad (3.30)$$

and therefore the eigenvalues of Q_3 are $\pm \sqrt{\lambda_{3E}}$ each one with multiplicity two. Making again the same calculations of the previous paragraph and introducing the momentum components longitudinal and transverse to the electric field, given by (3.20) with \vec{B} substituted by \vec{E} , we finally arrive to the effective action

$$S_{\text{eff}}[A] = -i \ln \det \left[\prod_{\eta=\pm 1} \left(p_0^2 - p_\ell^2 - m^2 - \left(p_t + \eta \frac{e\mu}{4m} E \right)^2 \right) \right]. \quad (3.31)$$

The effective Lagrangian, therefore, can be written as

$$\mathcal{L}_{\text{eff}} = \frac{i}{8\pi^3} \sum_{\eta=\pm 1} \int_0^\infty \frac{ds}{s} \int_{-\infty}^\infty dp_0 \int_{-\infty}^\infty dp_\ell \int_0^\infty p_t dp_t$$

$$\cdot \exp \left[-is \left(-p_0^2 + p_\ell^2 + m^2 + \left(p_t + \eta \frac{e\mu E}{4m} \right)^2 - i\epsilon \right) \right] \quad (3.32)$$

Introducing

$$\gamma_E = (|e\mu| E)/(4m^2), \quad z = m^2 s, \quad (3.33)$$

and going through steps analogous to those of item (i), we arrive to the normalized effective Lagrangian which vanishes for a vanishing electric field:

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & \frac{1}{2}E^2 - \frac{m^4}{8\pi^2} \int_0^\infty \frac{dz}{z^3} \exp(-z) \left[\exp(-\gamma_E^2 z) - 1 \right] \\ & - \frac{m^4 \gamma_E^2}{8\pi^2} \int_0^\infty \frac{dz}{z^2} \exp(-z) \int_{-1}^1 du \exp(-\gamma_E^2 u^2 z) \end{aligned} \quad (3.34)$$

The expression (3.34) is again singular for $z = 0$. We can subtract the leading divergent terms obtaining a minimally regularized effective Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & \frac{1}{2}E^2 - \frac{m^4}{8\pi^2} \int_0^\infty \frac{dz}{z^3} \exp(-z) \cdot \\ & \left[\exp(-\gamma_E^2 z) - 1 + 2\gamma_E^2 z \int_0^1 du [\exp(-\gamma_E^2 u^2 z)] - \gamma_E^2 z + \frac{1}{6}\gamma_E^4 z^2 \right] + f.t. \end{aligned} \quad (3.35)$$

where *f.t.* indicates the presence of possible finite terms to be determined by a specific renormalization prescription.

4 Finite temperature extension.

In this final section we will extend our analysis to finite temperatures: we shall therefore calculate the partition functions for the fermion systems studied in the previous section.

We recall that the finite temperature extension of the theory is done for the stationary case and for uniform fields we can substitute V_4 with βV , where $V = L_1 L_2 L_3$ is the spatial volume and $\beta = 1/T$ in units in which the Boltzmann constant k is taken unity. The energy is then discretized to the Matsubara frequencies $p_0 = (i\pi/\beta)(2j+1)$ and the integral $\int dp_0/(2\pi)$ is therefore changed into $(i/\beta) \sum_{j=-\infty}^{+\infty}$ [18]. The sum is easily dealt with by using the Poisson relation that, for the fermionic case [21], reads

$$\sum_{j=-\infty}^{+\infty} \delta \left(p_0 - \frac{\pi}{\beta} (2j+1) \right) = \beta \sum_{j=-\infty}^{+\infty} (-1)^j \exp(ij\beta p_0). \quad (4.1)$$

In order to show the subsequent steps of the calculation, take first a Dirac particle in a magnetic field whose the effective action is $S_{\text{eff}}[A] = J_+ + J_-$ with J_\pm given in (2.23). We have

$$\mathcal{L}_{\text{eff}} = i \int_0^\infty \frac{ds}{s} \sum_{\pm, n=0}^\infty \exp(-isM_\pm^2) i \sum_j (-1)^j \int_{-\infty}^{+\infty} \frac{dp_0}{2\pi} \int_{-\infty}^{+\infty} \frac{dp_z}{2\pi}$$

$$\exp(i(-p_0^2 - p_z^2)s) \exp(ij\beta p_0) \frac{qB}{2\pi} \exp(-iqB(2n+1)s) \quad (4.2)$$

Integrating over p_0 and p_z , summing over \pm and n and finally rotating the integration path of the variable s as in the zero temperature case, we get

$$\mathcal{L}_{\text{eff}} = \frac{1}{8\pi^2} \int_0^{+\infty} \frac{ds}{s^3} \exp(-m^2 s) (sqB) \coth(qBs) \sum_{j=-\infty}^{+\infty} (-1)^{j+1} \exp\left(\frac{-j^2\beta^2}{4s}\right) \quad (4.3)$$

Since the free Helmholtz energy density f is defined as

$$f = \frac{F}{V} = -\frac{\ln Z}{\beta V} = -\mathcal{L}_{\text{eff}}, \quad (4.4)$$

from (4.4) we clearly see that the zero temperature contribution is simply obtained by taking $j = 0$, so that we can write f as

$$f = \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} \exp(-sm^2) (sqB) \left(\coth(sqB) - \frac{1}{3} (sqB)^2 - 1 \right) + \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} \exp(-sm^2) (sqB) \coth(sqB) \sum_{j \neq 0} (-1)^j \exp\left(\frac{-j^2\beta^2}{4s}\right) \quad (4.5)$$

where the zero temperature part is the one we have previously studied, neglecting the tree level contribution $-(1/2)B^2$. In the temperature dependent term we make the expansion

$$x \coth(x) = x \left[\sum_{n=0}^{\infty} (\exp(-2nx)) + \sum_{n=0}^{\infty} (\exp(-2(n+1)x)) \right] \quad (4.6)$$

and we recall the integral representation for the K Bessel functions [22]:

$$K_\nu(2(z\zeta)^{\frac{1}{2}}) = \frac{1}{2} \left(\frac{\zeta}{z}\right)^{\nu/2} \int_0^\infty dt \exp\left(-zt - \frac{\zeta}{t}\right) t^{-\nu-1} \quad (4.7)$$

We then see that in (4.5) we are left with a series in terms of the Bessel function K_1 . By means of the further integral representation

$$K_1(z) = \int_0^\infty dt \exp(-z\sqrt{1+t^2}) \quad (4.8)$$

we get

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{j} u K_1(j\beta u) = - \int_0^\infty dt \ln\left(1 + \exp(-\beta\sqrt{t^2 + u^2})\right), \quad (4.9)$$

and we finally obtain the following expression for the free Helmholtz energy density:

$$f = \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} \exp(-sm^2) (sqB) \left(\coth(sqB) - \frac{1}{3} (sqB)^2 - 1 \right) +$$

$$-\frac{2qBT}{\pi} \sum_{n=0}^{\infty} \int_0^{\infty} \frac{dp_z}{2\pi} \left[\ln \left(1 + \exp(-\beta \sqrt{p_z^2 + m^2 + 2nqB}) \right) + \ln \left(1 + \exp(-\beta \sqrt{p_z^2 + m^2 + 2(n+1)qB}) \right) \right] \quad (4.10)$$

We will now apply these ideas to the fermions with anomalous magnetic moment we have treated in Section 3 in order to calculate the close form for their free energy density.

(i) The free energy for a spinor with a.m.m. in a magnetostatic uniform field. Starting from (3.22) and following the steps previously outlined, a simple calculation leads to the following expression for the effective Lagrangian:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(B, 0) + \mathcal{L}_{\text{eff}}(B, T) \quad (4.11)$$

where $\mathcal{L}_{\text{eff}}(B, 0)$ is given by (3.23) or, better, by its regularized form (3.26). The second term

$$\mathcal{L}_{\text{eff}}(B, T) = \sum_{\eta=\pm 1} \sum_{j=1}^{\infty} \frac{(-)^{j+1}}{4\pi^2} \int_0^{\infty} \frac{ds}{s^2} \int_0^{\infty} p_t dp_t \exp \left(\frac{j^2 \beta^2}{4s} - s \left(\sqrt{p_t^2 + m^2} + \eta \frac{e\mu B}{4m} \right)^2 \right), \quad (4.12)$$

where the sum in j has been arranged from one to infinity due to parity, represents the contribution at non vanishing temperature. In view of the relation (4.7) we can write

$$\mathcal{L}_{\text{eff}}(B, T) = \sum_{\eta=\pm 1} \sum_{j=1}^{\infty} \frac{(-)^{j+1}}{\beta \pi^2} \int_0^{\infty} p_t dp_t K_1 \left[j\beta \left(\sqrt{p_t^2 + m^2} + \eta \frac{e\mu B}{4m} \right) \right] \cdot \left(\sqrt{p_t^2 + m^2} + \eta \frac{e\mu B}{4m} \right) \quad (4.13)$$

By using the representation (4.8) of the Bessel function, we can give the final form for the temperature part of free energy density of the spinor with a.m.m. in a magnetostatic field, namely

$$f(B, T) = \frac{(-1)}{2\pi^2 \beta} \sum_{\eta=\pm 1} \int_0^{\infty} p_t dp_t \int_{-\infty}^{\infty} dp_{\ell} \ln \left[1 + \exp \left(-\beta \sqrt{p_{\ell}^2 + \left(\sqrt{p_t^2 + m^2} + \eta \frac{e\mu B}{4m} \right)^2} \right) \right] \quad (4.14)$$

(ii) The free energy for a spinor with a.m.m. in an electrostatic field. We now start from (3.32) and make the usual substitutions described above, arriving at

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(E, 0) + \mathcal{L}_{\text{eff}}(E, T) \quad (4.15)$$

where $\mathcal{L}_{\text{eff}}(E, 0)$ is given by the regularized expression (3.35) and

$$\mathcal{L}_{\text{eff}}(E, T) = \sum_{\eta=\pm 1} \sum_{j=1}^{\infty} \frac{(-)^{j+1}}{\beta \pi^2} \int_0^{\infty} p_t dp_t K_1 \left[j\beta \sqrt{m^2 + \left(p_t + \eta \frac{e\mu E}{4m} \right)^2} \right]$$

$$\cdot \sqrt{m^2 + \left(p_t + \eta \frac{e\mu E}{4m}\right)^2} \quad (4.16)$$

Using once again (4.7)-(4.9), we eventually get

$$f(E, T) = -\frac{1}{2\pi^2\beta} \sum_{\eta=\pm 1} \int_0^\infty p_t dp_t \int_{-\infty}^\infty dp_\ell \ln \left[1 + \exp \left(-\beta \sqrt{p_\ell^2 + m^2 + \left(p_t + \eta \frac{e\mu E}{4m}\right)^2} \right) \right] \quad (4.17)$$

which represents the counterpart of (4.14) for the electrostatic case.

To conclude we would observe that in this paper we made a concrete use of a series of our previous results [1–3] in which we pointed out that the Foldy-Wouthuysen transformations can be used as an efficient calculation tool for some physical systems. We previously proved [2] that these are the cases in which the FWT can be done exactly. In this paper we have then applied the method to the case of relativistic fermions with anomalous magnetic moment in uniform magnetostatic and electrostatic fields. We have thus been able to produce a closed form for the effective action both at zero and finite temperature and therefore the free Helmholtz energy of such systems.

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